Mathematics II

(English course)

Second semester, 2012/2013

Exercises (3)

1. Sketch the level curves for the functions $z = f(x, y)$, with $f(x, y)$ given by the following expressions:

   (a) $f(x, y) = y + x - x^2$;

   (b) $f(x, y) = \frac{x^2}{4} + \frac{y^2}{16}$;

   (c) $f(x, y) = (x - 1)(y + 2)$;

   (d) $f(x, y) = \frac{xy}{x^2+y^2}$;

   (e) $f(x, y) = \frac{xy^2}{x^2+y^4}$;

   (f) $f(x, y) = \frac{x+y}{x-y}$;

   (g) $f(x, y) = \frac{x^2+y^2}{x^2-y^2}$.

2. For each of the following expressions, find the maximal domain in $\mathbb{R}^2$ where they define a real function. Sketch their representations in the plane.

   (a) $f(x, y) = \left(y + \sin \frac{1}{x}\right)^{-1}$;

   (b) $f(x, y) = \ln (\sin x) + y^{-\frac{1}{2}}$;

   (c) $f(x, y) = \frac{\sqrt{4-x^2-y^2}}{1+\ln x}$;

   (d) $f(x, y) = \sqrt{\ln (x^2 + 2xy + y^2)}$;

   (e) $f(x, y) = \frac{\sqrt[3]{1-xy}}{\ln(x^2)-\ln(x+y)}$;

   (f) $f(x, y) = \frac{\sqrt{4-|x|-|x+y|}}{1-\sqrt{|x|+|y|}}$.

3. Which of the following sets are open? Which ones are closed? Which

ones are bounded?

$$\begin{aligned}
A &= \left\{(x,y)\in\mathbb{R}^2 : xy > 1\right\};\\
B &= \left\{(x,y)\in\mathbb{R}^2 : xy \le 1\right\};\\
C &= \left\{(x,y)\in\mathbb{R}^2 : (x-1)^2 + 4y^2 \le 1\right\};\\
D &= \left\{(x,y)\in\mathbb{R}^2 : x \ge 0,\ y = x + \frac{1}{n}, n\in\mathbb{N}\right\};\\
E &= \left\{\left(\frac{1}{n}, \frac{n-1}{n}\right) : n\in\mathbb{N}\right\};\\
F &= \left\{\left(\frac{n+1}{n}, \frac{(-1)^n n^2}{n^2+1}\right) : n\in\mathbb{N}\right\}.
\end{aligned}$$

4. For each of the sets above, find its interior, boundary and closure.

5. Which of the following sets are open? Which ones are closed?

$$\begin{aligned}
A &= \bigcup_{n=1}^{\infty}\left\{(x,y)\in\mathbb{R}^2 : \left(x-\frac{2}{n}\right)^2 + y^2 \le \frac{1}{n^3}\right\};\\
B &= \bigcup_{n=1}^{\infty}\left\{(x,y)\in\mathbb{R}^2 : \left(x-\frac{3}{n}\right)^2 + \left(y+\frac{1}{n}\right)^2 < \frac{1}{n^3}\right\};\\
C &= \bigcap_{n=1}^{\infty}\left\{(x,y)\in\mathbb{R}^2 : \left(x+\frac{1}{n}\right)^2 + \left(y-\frac{1}{n}\right)^2 < 1+\frac{6}{n}\right\};\\
D &= \bigcap_{n=1}^{\infty}\left\{(x,y)\in\mathbb{R}^2 : x+y \ge -\frac{x^2+\sqrt{1+x^2+y^2}}{n}\right\}.
\end{aligned}$$

6. For each of the following sequences, find its limit (provided the sequence is convergent).

   (a) $a_n = \left(\frac{n^2+n}{\sqrt{n^5+1}-1}, \ln\left(\frac{n+1}{n}\right)^n\right)$;

   (b) $a_n = \left(n^2+1, \sqrt{\left(\frac{3n+1}{n+1}\right)^n}\right)$;

   (c) $a_n = \left(n\left(e^{\frac{1}{n+4}}-1\right), \frac{n^2+1}{n}\sin\frac{\pi}{n}\right)$;

   (d) $a_n = \left(\sqrt{n}\left(\sqrt{2n+1}-\sqrt{2n-1}\right), \left(1-\frac{5}{n}\right)^n, n^2\left(\cos\frac{2}{n}-1\right)\right)$;

   (e) $a_n = \left(\frac{\sqrt{n}+1}{2n-1}, \frac{n+(-1)^n n^2}{(n+1)^2}, \sin\frac{1}{n}\right)$;

7. Compute the following limits or show that they don't exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{(x+y)^2}$;

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{(x+y)^2+y^2}$;

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{(x+y)^2}$;

(d) $\lim\limits_{(x,y)\to(0,0)} \frac{xy+y}{x^2+2y^2}$;

(e) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, where

$$f(x,y) = \begin{cases} \frac{x}{y}\sin(x+y), & \text{for } x > 0 \text{ and } y > 0, \\ 0, & \text{for } x \le 0 \text{ or } y \le 0; \end{cases}$$

(f) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, $\lim\limits_{(x,y)\to(0,1)} f(x,y)$, and $\lim\limits_{(x,y)\to(2,2)} f(x,y)$, where

$$f(x,y) = \begin{cases} x + y - \sqrt{xy}, & \text{for } x > 0 \text{ and } y > 0, \\ 0, & \text{for } x \le 0 \text{ or } y \le 0; \end{cases}$$

(g) $\lim\limits_{(x,y)\to(2,1)} \frac{(x-2)^2 \ln y}{(x-2)^2+(\ln y)^2}$.